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Tensor-Based Abduction in Horn Propositional Programs

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Why Linear Algebra?

- AI software is moving towards GPU-based solutions
- Optimized for matrix/tensor multiplication
- Highly-Parallel computations
- Develop algorithms for GPU

Abduction

- Logical Inference through Explanations

$$\langle P, g, Ab \rangle$$

- Assume P is a Horn Logic Program
- Observation g
- Find $\Delta \subseteq Ab$ such that $g \in LHM(P \cup \Delta)$
- $P \cup \Delta$ is consistent

Background: Embedding Atoms[1]

$$g \leftarrow p \ \wedge \ q$$
$$p \leftarrow q$$
$$q \leftarrow$$
$$\leftarrow g$$

$$\vec{v}_{\perp} = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} \quad \vec{v}_{\top} = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}$$

$$\vec{v}_g = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} \quad \vec{v}_p = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} \quad \vec{v}_q = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

$$\vec{v}_{\{g,q\}} = \begin{bmatrix}0\\1\\1\\0\\1\end{bmatrix}$$

1. Sakama, C., Inoue, K., Sato, T.: Linear Algebraic Characterization of Logic Programs. In: KSEM. Lecture Notes in Computer Science, vol. 10412, pp. 520-533. Springer (2017)

Background: Embedding Programs[1]

$g \leftarrow p \wedge q$
$p \leftarrow q$
$q \leftarrow$
$\leftarrow g$

$$D^P = \begin{array}{cc} & \begin{array}{ccccc} \bot & \top & g & p & q \end{array} \\ & \left[\begin{array}{ccccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \end{array} \end{array}$$

Immediate Consequence via matrix multiplication:

$$J = T_P(I) \cup I \quad \Leftrightarrow \quad \vec{v}_J = H_1(D^P \cdot \vec{v}_I) \qquad H_1(x) = \begin{cases} 0, & x < 1 \\ 1, & x \geq 1 \end{cases}$$

1. Sakama, C., Inoue, K., Sato, T.: Linear Algebraic Characterization of Logic Programs. In: KSEM. Lecture Notes in Computer Science, vol. 10412, pp. 520-533. Springer (2017)

Abduction - Main Idea

Invert implications:

$$g \leftarrow p \land q$$
$$g \leftarrow q$$
$$g \leftarrow t$$

$$\begin{tikzcd} & \{p, q\} \\ \{g\} \arrow[ur] \arrow[r] \arrow[dr] & \{q\} \\ & \{t\} \end{tikzcd}$$

Potential solutions: $\{\{p, q\}, \{q\}, \{t\}, \{g\}\}$

Tensor Embedding

$$\begin{array}{cc} \begin{array}{cccccc} \bot & \top & g & p & q & t \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

$$g \to p \wedge q$$

$$A_{::1} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

$$g \rightarrow p \wedge q$$

$$A_{::2} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

$$g \rightarrow q$$

$$A_{::3} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

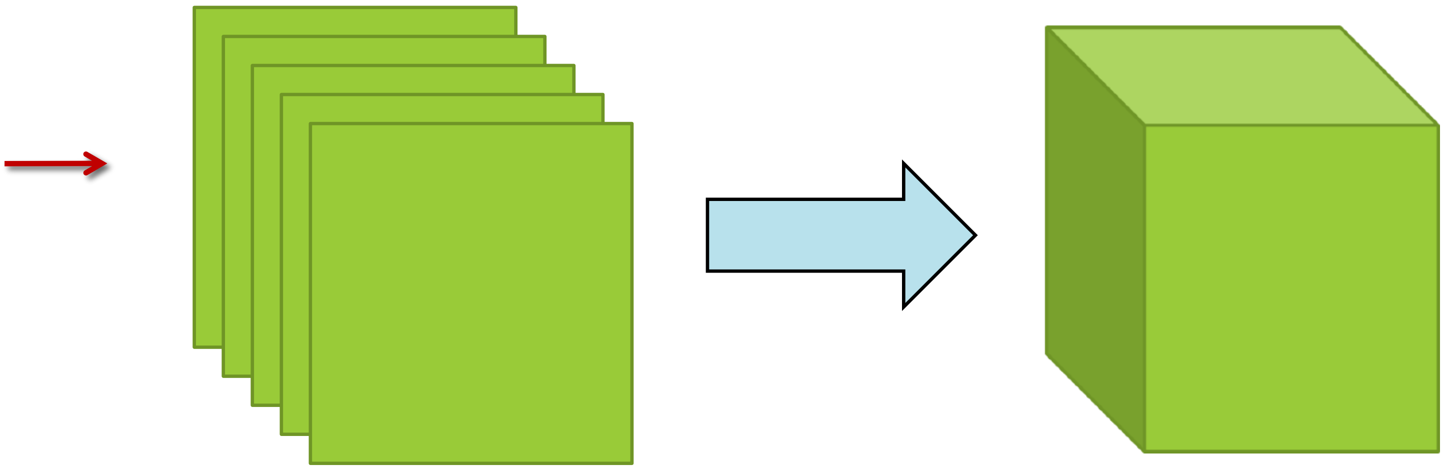
$$g \rightarrow t$$

$$A_{::4} = \begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

Keep $\{g\}$

Frontal Slices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



$A_{::k}$

Third-Order Tensor A

Abductive Step

$$H_1\left(A \times_2 \begin{bmatrix}0\\1\\1\\0\\0\\0\end{bmatrix}\right) = \begin{array}{c} \begin{bmatrix}0 & 0 & 0 & 0\\1 & 1 & 1 & 1\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\1 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix} \end{array} \begin{array}{c}\bot\\ \top\\ g\\ p\\ q\\ t\end{array}$$

$$\{g\} \quad \Rightarrow \quad \{\{p,q\},\{q\},\{t\},\{g\}\}$$

Inconsistencies

$$g \leftarrow p \ \wedge \ q$$
$$g \leftarrow q$$
$$g \leftarrow \ t$$
$$\leftarrow t$$

So far: $\begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{matrix} \bot \\ \top \\ g \\ p \\ q \\ t \end{matrix} \end{array}$

Idea: Compute $LHM(P \cup \Delta)$ for each column

Inconsistencies

$$
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\underset{LHM}{\Rightarrow}
\begin{array}{cc}
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} &
\begin{matrix} \bot \\ \top \\ g \\ p \\ q \\ t \end{matrix}
\end{array}
$$

Inconsistencies

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\underset{LHM}{\Rightarrow}
\begin{array}{cc}
\begin{array}{cccc} & & \text{Inconsistent!} & \end{array} & \\
\begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
&
\begin{array}{c}
\bot \\
\top \\
g \\
p \\
q \\
t
\end{array}
\end{array}
$$

Inconsistencies

Inconsistent!

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underset{LHM}{\Rightarrow} \begin{array}{cc} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ t \end{array} \end{array}$$

Remove inconsistencies, duplicates, and continue to Abductive Step…

Abducibles

- $\Delta \subseteq Ab$ if and only if:

$$\vec{v}_\Delta \times \vec{v}_{Ab} = \vec{v}_\Delta$$

- Post-Filtering
- If Abducibles are not defined, then $\Delta \subseteq Ab$ implies $H_1(A^P \times_2 \vec{v}_\Delta) = \vec{v}_\Delta$ after duplicates are removed

(Filter during run)

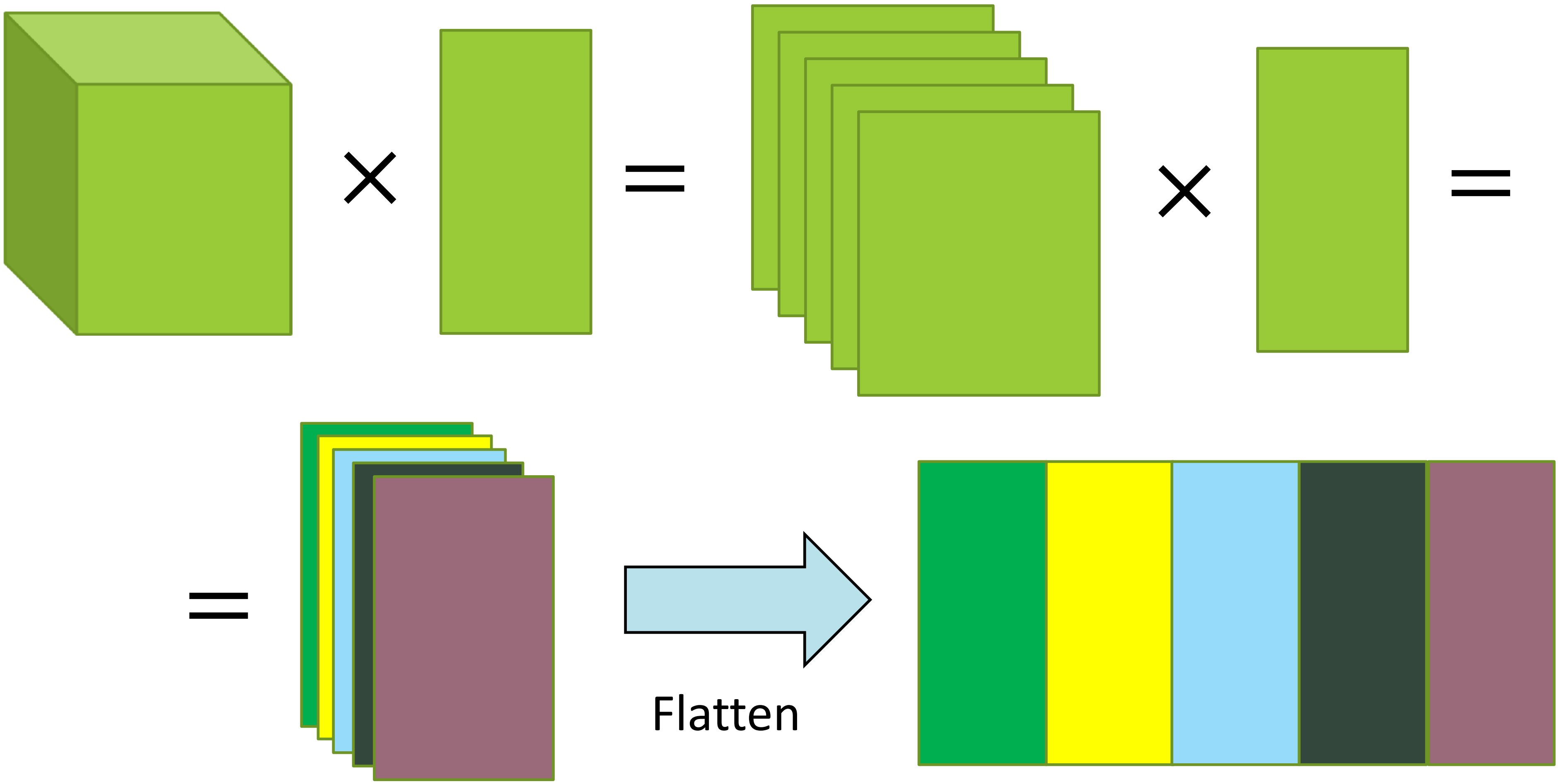
Discussion and Future Work

- Proof of correctness has been completed
- An unoptimised implementation has been made

Future Work:

- Clauses with negation
- First-Order Predicate Logic
- Optimisation and Scalability Testing

Tensor Multiplication



Multiple Definitions

$$g \leftarrow p \ \land \ q$$
$$g \leftarrow p \ \land \ r$$

$$H\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{array}{cc} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{array}{c} \bot \\ \top \\ g \\ p \\ q \\ r \end{array} \end{array}$$

Introduce auxiliary variables:

$$g_1 \leftarrow p \ \land \ q \qquad\qquad g \leftarrow g_1$$
$$g_2 \leftarrow p \ \land \ r \qquad\qquad g \leftarrow g_2$$