The Complexity and Generality of Learning Answer Set Programs

(AIJ 2018)

Mark Law, Alessandra Russo and Krysia Broda

September 2, 2018
Several ILP frameworks have been proposed to learn ASP:

- In $\text{ILP}_b$ (resp $\text{ILP}_c$) at least one (resp every) answer set of $B \cup H$ must cover the (atom) examples.
- In $\text{ILP}_{\text{LAS}}$ examples are partial interpretations and a combination of $\text{ILP}_b$ and $\text{ILP}_c$ can be expressed.
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This paper asks two fundamental questions:

- What class of ASP programs can each framework learn?
- Is there any (complexity) price paid by the more general frameworks?
ILP under the Answer Set Semantics

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- This paper asks two fundamental questions:
  - What class of ASP programs can each framework learn?
  - Is there any (complexity) price paid by the more general frameworks?

- In the paper we also consider $ILP_{sm}$, $ILP_{LOAS}$ and $ILP_{context}$. 
Definition 1

A learning framework \( \mathcal{F} \) can distinguish \( H_1 \) from \( H_2 \) wrt \( B \) iff there is at least one task \( T_\mathcal{F} = \langle B, E_\mathcal{F} \rangle \) such that \( H_1 \in \mathcal{F}(T_\mathcal{F}) \) and \( H_2 \notin \mathcal{F}(T_\mathcal{F}) \).
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- \( D^1_{\mathcal{F}} \) is the set of tuples \( \langle B, H_1, H_2 \rangle \) such that \( \mathcal{F} \) can distinguish \( H_1 \) from \( H_2 \) wrt \( B \).
One-to-one Distinguishability

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Let $B = \emptyset$, $H_1 = \{p.\}$ and $H_2 = \{0\{p\}1.\}$.

- $\langle B, H_1, H_2 \rangle$ is not in $D_1^1(ILP_b)$.
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$E^+ = \{p\}$ \hspace{2cm} $E^- = \emptyset$

$H_2 \in ILP_b(\langle B, \{p\}, \emptyset \rangle)$. 

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$H_2 \in ILP_b(\langle B, \emptyset, \{p\} \rangle)$ but $H_1 \notin ILP_b(\langle B, \emptyset, \{p\} \rangle)$. 
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A learning framework \( \mathcal{F} \) can *distinguish* \( H_1 \) from \( H_2 \) wrt \( B \) iff there is at least one task \( T_\mathcal{F} = \langle B, E_\mathcal{F} \rangle \) such that \( H_1 \in \mathcal{F}(T_\mathcal{F}) \) and \( H_2 \notin \mathcal{F}(T_\mathcal{F}) \).

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- \( \langle B, H_1, H_2 \rangle \) is not in \( D^1_1(ILP_b) \).
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$E^+ = \{p\}$ \hspace{2cm} $E^- = \emptyset$

$H_1 \in ILP_c(\langle B, \emptyset, \{p\} \rangle)$ but $H_2 \notin ILP_c(\langle B, \emptyset, \{p\} \rangle)$. 

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### One-to-one Distinguishability Conditions

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<tr>
<th>Framework $\mathcal{F}$</th>
<th>Sufficient/necessary condition for $\langle B, H_1, H_2 \rangle$ to be in $\mathcal{D}_1^1(\mathcal{F})$</th>
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<td>$ILP_{b}$</td>
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<td>$ILP_{context}^{LOAS}$</td>
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<td>$(\exists C \in \text{ASP}^{\text{ch}} \text{ st } \text{ord}(B \cup H_1 \cup C) \neq \text{ord}(B \cup H_2 \cup C))$</td>
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- Neither $ILP_b$ of $ILP_{sm}$ can distinguish $H \cup C$ from $H$ for any constraint $C$ and any $H$ – in practice, neither $ILP_b$ nor $ILP_{sm}$ can learn constraints.
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- $ILP_{LAS}$ can distinguish any two hypotheses, so long as they have different answer sets (when combined with $B$).
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- $ILP_{LOAS}^{\text{context}}$ can distinguish any two hypotheses, so long as they are not strongly equivalent (when combined with $B$).
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$\mathcal{D}_1^1(ILP_b) = \mathcal{D}_1^1(ILP_{sm}) \subset \mathcal{D}_1^1(ILP_{LAS}) \subset \mathcal{D}_1^1(ILP_{LOAS}) \subset \mathcal{D}_1^1(ILP_{context}^{LOAS})$

$\mathcal{D}_1^1(ILP_c) \subset \mathcal{D}_1^1(ILP_{LAS})$
### Definition 2

For a framework $\mathcal{F}$, $D_m^1(\mathcal{F})$ is the set of tuples $\langle B, H, \{H_1, \ldots, H_n\} \rangle$ such that there is a task $T_\mathcal{F}$ which distinguishes $H$ from each $H_i$ with respect to $B$. 
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For a framework $\mathcal{F}$, $D_m^1(\mathcal{F})$ is the set of tuples $\langle B, H, \{H_1, \ldots, H_n\}\rangle$ st there is a task $T_\mathcal{F}$ which distinguishes $H$ from each $H_i$ with respect to $B$.

Let $B = \emptyset$, $H = \{\text{heads, tails}1.\}$, $H'_1 = \{\text{heads.}\}$, $H'_2 = \{\text{tails.}\}$

- $\langle B, H, H'_1 \rangle \in D_1^1(\text{ILP}_b)$ and $\langle B, H, H'_2 \rangle \in D_1^1(\text{ILP}_b)$
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Let $B = \emptyset$, $H = \{1\{\text{heads}, \text{tails}\}1.\}$, $H'_1 = \{\text{heads}\}$, $H'_2 = \{\text{tails}\}$.

- $\langle B, H, H'_1 \rangle \in \mathcal{D}_1^1(\text{ILP}_b)$ and $\langle B, H, H'_2 \rangle \in \mathcal{D}_1^1(\text{ILP}_b)$
- $\langle B, H, \{H'_1, H'_2\}\rangle \not\in \mathcal{D}_m^1(\text{ILP}_b)$
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\[
\mathcal{D}_m^1(ILP_b) \subset \mathcal{D}_m^1(ILP_{sm}) \subset \mathcal{D}_m^1(ILP_{LAS}) \subset \mathcal{D}_m^1(ILP_{LOAS}) \subset \mathcal{D}_m^1(ILP_{context})
\]

\[
\mathcal{D}_m^1(ILP_c) \subset \mathcal{D}_m^1(ILP_{LAS})
\]
Many-to-many Distinguishability

Definition 3

For a framework \( \mathcal{F} \), \( D_m^m(\mathcal{F}) \) is the set of tuples \( \langle B, S_1, S_2 \rangle \), st there is a task \( T_{\mathcal{F}} \) with background \( B \), st \( S_1 \subseteq ILP_{\mathcal{F}}(T_{\mathcal{F}}) \) and \( S_2 \cap ILP_{\mathcal{F}}(T_{\mathcal{F}}) = \emptyset \).
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For a framework $\mathcal{F}$, $\mathcal{D}_m^m(\mathcal{F})$ is the set of tuples $\langle B, S_1, S_2 \rangle$, st there is a task $T_{\mathcal{F}}$ with background $B$, st $S_1 \subseteq ILP_{\mathcal{F}}(T_{\mathcal{F}})$ and $S_2 \cap ILP_{\mathcal{F}}(T_{\mathcal{F}}) = \emptyset$.

\[
\mathcal{D}_m^m(ILP_b) \subset \mathcal{D}_m^m(ILP_{sm}) \subset \mathcal{D}_m^m(ILP_{LAS}) \subset \mathcal{D}_m^m(ILP_{LOAS}) \subset \mathcal{D}_m^m(ILP_{LOAS}^{\text{context}})
\]

\[
\mathcal{D}_m^m(ILP_c) \subset \mathcal{D}_m^m(ILP_{LAS})
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## Complexity

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Conclusion

- We have introduced three new measures of the generality of a learning framework.

- For each of the three measures:
  \[ D(\text{ILP}_b) \subseteq D(\text{ILP}_{sm}) \subseteq D(\text{ILP}_{LAS}) \subseteq D(\text{ILP}_{LOAS}) \subseteq D(\text{ILP}_{context}) \]
  \[ D(\text{ILP}_c) \subseteq D(\text{ILP}_{LAS}) \]

- There is no price to be paid (in terms of complexity) for the gain in generality of \( \text{ILP}_{LOAS}^{context} \) over \( \text{ILP}_c \).

- \( \text{ILP}_b \) and \( \text{ILP}_{sm} \) are of lower complexity, but are less general than \( \text{ILP}_{LAS} \).
Backup Slides
In the paper, we proved that if for any two \( \mathcal{F} \) tasks \( T_1, T_2 \) there is a task \( T_3 \) such that \( ILP_{\mathcal{F}}(T_3) = ILP_{\mathcal{F}}(T_1) \cap ILP_{\mathcal{F}}(T_2) \) then:

\[
D_{m}^{1}(\mathcal{F}) = \left\{ \langle B, H, \{H_1, \ldots, H_n\}\rangle \mid \langle B, H, H_1\rangle \in D_{1}^{1}(\mathcal{F}), \ldots, \langle B, H, H_n\rangle \in D_{1}^{1}(\mathcal{F}) \right\}.
\]

In \( ILP_{\text{LAS}} \), \( T_3 \) can be constructed as \( \langle B, E_1^+ \cup E_2^+, E_1^- \cup E_2^- \rangle \).

This property holds for every framework (in the paper) other than \( ILP_b \).

\[
D_{m}^{1}(ILP_b) \subset D_{m}^{1}(ILP_{\text{sm}}) \subset D_{m}^{1}(ILP_{\text{LAS}}) \subset D_{m}^{1}(ILP_{\text{LOAS}}) \subset D_{m}^{1}(ILP_{\text{context}}) \\
D_{m}^{1}(ILP_c) \subset D_{m}^{1}(ILP_{\text{LAS}})
\]
Brave Induction cannot learn constraints

- Let $H$ be a hypothesis and $C$ be a constraint.

- For any $T = \langle B, E^+, E^- \rangle$ st $H \cup C \in ILP_b(T)$, there is an $A \in AS(B \cup H \cup C)$ st $E^+ \subseteq A$ and $E^- \cap A = \emptyset$.

  Any such $A$ is also an answer set of $B \cup H$.

- Hence $ILP_b$ cannot distinguish $H \cup C$ from $H$ (wrt any background knowledge).

- In practice this means that $ILP_b$ cannot learn constraints.
Other notion of generality

- (De Raedt 1997) defined generality in terms of reductions. \( F_1 \) is said to be more general than \( F_2 \) iff \( F_2 \to_r F_1 \) and \( F_1 \not\to_r F_2 \).

- These reductions allowed the background knowledge \( B \) to be modified in the reduction, whereas distinguishability does not.

- In the paper we define strong reductions which force the background knowledge to be the same and show that \( F_1 \to_{sr} F_2 \) if and only if \( D^m_m(F_1) \subseteq D^m_m(F_2) \).

- Other than the restriction on the background knowledge, distinguishability also allows for fine grained comparisons of frameworks which are incomparable under reductions and strong reductions.